September 10, 2009

Dear Professor Langlands,

I will try to answer your mathematical questions about orbifolds and the Harer-Zagier formula. For my own part, I do not know about stacks, knowing them only as being sort of like orbifolds. Forgive me if my exposition is too pedestrian - I am sure you have seen much of what I say below, but I decided to err in the direction of saying too much rather than too little so as to avoid losing you. On the other hand, there are places where I have been sketchy, either to avoid a lengthy digression or on account of my own ignorance.

Orbifolds

The first point to make is that I have the impression that just as there are many categories of manifolds (topological, differentiable, complex, piecewise linear, Riemannian, hyperbolic, etc.) there are correspondingly many categories of orbifolds, each of which generalizes the corresponding "manifold" notion. How the definition of each category of manifold should be modified to be the corresponding orbifold category may be a nontrivial task and is not one that I have the background to address. Also, I am only peripherally familiar with the general notion of an orbifold at all: my familiarity is with the subclass of orbifolds that are quotients of a contractible space by a discrete group acting properly discontinuously on the space.

A second point to make is that an orbifold is what Satake called a "V-manifold."

Let X be a topological manifold which is *contractible*. This means that X is homotopic to a point, equivalently, the k-th homotopy group of X is trivial for every integer $k \geq 0$ (here the nontrivial direction is a theorem of Whitehead). In practice X is homeomorphic to $\mathbb{R}^n$ for some n, at least if X is without boundary. Suppose G is a group acting on X and that for every $g \in G$, the map $g : X \longrightarrow X$ is a homeomorphism. We say that G acts *properly discontinuously* on X if for every compact set $K \subset X$, the set of $g \in G$ such that $\{gK \cap K \neq \emptyset\}$ is finite. We say that G *acts freely* on X if for all $p \in X$, the only element of G that fixes p is the identity of G.

Then it is a basic theorem (e.g. Proposition 3.5.7 of [15]) that X/G is a (topological) manifold with fundamental group $\pi_1(X/G) \cong G$ if and only if G acts properly discontinuously and freely on X. If X has some sort of additional structure (complex structure, differential structure, etc.) then X/G inherits this structure.

The condition that G act properly discontinuously on X can be weakened to accommodate broader classes of "discrete groups" at the cost of X/G failing to be a manifold (perhaps the first thing that is lost is Hausdorffness). But in practice the natural action of the discrete groups that one is most interested in is properly discontinuous (that is

"discrete in a weak sense" + "natural" $\longrightarrow$ "discrete in a strong sense" = properly discontinuous) and it is useful to assume that the action is properly discontinuous to avoid pathologies.

However, the condition that G act freely is much less frequently present for natural actions of groups on spaces. One can show that if G has nontrivial torsion, the torsion elements have fixed points and so that the action of G on X is not free. For example, if $G = \mathrm{PSL}_2(\mathbb{Z})$ acts on X = {the upper half plane $\mathbb{H}$} by linear fractional transformations, then some nontrivial elements of G fix points: namely $z \longrightarrow -1/z$ which fixes i and $z \longrightarrow 1 - 1/z$ which fixes $\omega = e^{\frac{2\pi i}{3}}$. For this example, the quotient $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ can be viewed as a topological manifold: it is homeomorphic to the once punctured sphere. But this point of view is unsatisfactory for some purposes: for example one does not have $\pi_1(\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z}))$ isomorphic to $\mathrm{PSL}_2(\mathbb{Z})$. If one wants to be able to see $\mathrm{PSL}_2(\mathbb{Z})$ in $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$, one should remember what the fixed points of $\mathrm{PSL}_2(\mathbb{Z})$ in $\mathbb{H}$ are, and the orders of the elements that stabilize them.

With this in mind, one can formally define the "orbifold fundamental group" $\tilde{\pi}_1(\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z}))$ to be usual fundamental group of the topological manifold underlying $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ with some base point other than i or ω, except that a single loop that encloses i and not ω should have the property that it has order 2 in $\tilde{\pi}_1(\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z}))$ and a single loop around ω that does not enclose i should have order 3 in $\tilde{\pi}_1(\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z}))$. One then gets $\tilde{\pi}_1(\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})) = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z})$ (that is, the free product of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$) and the latter is well known to be a presentation of $\mathrm{PSL}_2(\mathbb{Z}$, so with this definition of orbifold fundamental group, we have $\tilde{\pi}_1(\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})) \cong \mathrm{PSL}_2(\mathbb{Z})$ as desired. In fact the isomorphism can be taken to be natural - to do this, one mimics the covering space theory for the fundamental group of a manifold.

The loop around i (resp. ω) is taken to have order 2 (resp. 3) because that is the order of the element of $\mathrm{PSL}_2(\mathbb{Z})$ that fixes it.

One can give the following ad hoc definition: if X is homeomorphic to $\mathbb{R}^n$ for some n, G acts properly discontinuously on X by homeomorphisms, and every element $g \in G$ other than the identity that fixes a point in X has finite order, then the *orbifold* X/G is the topological space X/G together with the marking of all points in X/G that are fixed by some element of G with the order of the element(s) that fix them. The set of marked points is then referred to as the *singular locus* of the orbifold X/G. One can define "orbifold fundamental group" in this setting (following the spirit of the "definition" given for $\mathrm{PSL}_2(\mathbb{Z})$ above, and this leads to the theorem that $\tilde{\pi}_1(X/G) \cong G$ (canonically).

If X is endowed with a smooth/complex/hyperbolic structure then X/G inherits a smooth/complex/hyperbolic structure at all points other than those on the singular locus. This is another reason to keep track of the singular points: you necessarily see them when you try to push additional structure on X down to X/G.

If one gives the "right" definition of orbifold (encapsulating a certain intuitive notion of a space with isolated singularities of prescribed type) then one finds that not every orbifold arises as X/G for some X and G as above. However, the orbifolds that I had in mind when I wrote to you are of the type X/G for some X and G as above.

Orbifold Euler Characteristic

Having said something about what an orbifold is, I can introduce the "orbifold Euler characteristic" of an orbifold of type X/G. Suppose M is an n dimensionalmanifold admitting a finite simplicial decomposition D, that is, a realization as a gluing of finitely many k-simplices for $1 \leq k \leq n$, with the gluing being an identification of k-dimensional hyperfaces to k-dimensional hyperfaces for each k. The Euler characteristic of X is then defined to be

$$\chi(M) = \Sigma(-1)^k \; \{\text{number of } \textit{distinct } k\text{-dimensional hyperfaces in } D\} \tag{0.1}$$

Euler (and Descartes before him) noticed that quantity does not depend on the finite triangulation of M, for M is a two-sphere. It was subsequently gradually realized that $\chi(M)$ is a topological invariant of any manifold. For a fairly complete account of the early history of the above formula see Imre Lakatos' *Proofs and Refutations* [8]. From a modern perspective, the invariance is thought of as a consequence of the theorem that

$$\chi(M) = \Sigma(-1)^k \; \{\text{rank of } k\text{-th singular homology group of } M\} \tag{0.2}$$

and the fact that singular homology groups of a topological space are topological invariants. Now, using (0.1) and the definition of finite cover, if N is a finite cover of M of degree d then

$$\chi(N) = d \cdot \chi(M).$$

This last formula is the trivial case of what's called the Riemann-Hurwitz formula. It serves as the basis for the definition of the "orbifold Euler characteristic" of a "good" orbifold.

Suppose X/G is an orbifold with the property that G has a finite index subgroup G' with the property that G' is torsion-free so that the singular locus of X/G' is empty and X/G' inherits the natural manifold structure from X. (In this case we say that X/G is a *good orbifold*.) Let $d = [G : G']$. If X/G is a manifold then d is the degree of the covering map $X/G' \longrightarrow X/G$; if X/G is not a manifold then the number of preimages of a point in X/G' is not well defined, but d is still the number of preimages of a generic point of X/G' in X/G. One defines the *orbifold Euler characteristic* of X/G by

$$\chi_{\text{orb}}(X/G) := (1/d) \cdot \chi(X/G')$$

It is easy to see that this definition is independent of the choice of G' using the fact that If H_1 and H_2 two finite index subgroups of G then $H_1 \cup H_2$ is of finite index in G as well.

Now is a good time to give some examples. Consider the disk $D = \{(x, y)|x^2+y^2 \leq 1\}$ in the plane. We can view this disk as a gluing of four distorted triangles - one for each quadrant of the plane. Doing so, we have $V = 5$ vertices, $E = 9$ edges, and $F = 4$ faces so $\chi(D) = V - E + F = 1$. Let Γ be the group generated by rotation through $\frac{\pi}{2}$ radians in the counterclockwise direction. Then the orbifold D/Γ has $\chi_{\mathrm{orb}}(D/\Gamma) = 1/4$ since Γ has order 4.

Though I don't know the formal statement of the theorem that I'm using next, one can compute the last quantity directly from D/Γ: D/Γ can be identified with one of the four triangles and so has 1 face, 3 edges and 3 vertices, only the vertex that has $(0, 0)$ as its preimage should be weighted by $1/4$ since $(0, 0)$ is a fixed point of order 4. So $V = 2 + 1/4$, $E = 3$, $F = 1$ and $\chi_{\mathrm{orb}}(D/\Gamma) = V - E + F = 1/4$.

Note that the topological Euler characteristic χ of the space D/Γ (given by forgetting the weighting on the singular locus, or equivalently χ defined by (0.2)) is 1, so that the orbifold Euler characteristic χ_{orb} and ordinary Euler characteristic χ of an orbifold are in general different. The subscript 'orb' on χ_{orb} is nonstandard. Usually in contexts involving orbifolds when people say "Euler characteristic" without qualification and when they write χ they mean orbifold Euler characteristic, but I will continue using χ_{orb} throughout this message for orbifold Euler characteristic.

For a second example, we can compute the $\chi_{\mathrm{orb}}(\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z}))$, either by taking a torsion-free finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ or a direct computation, weighting the simplices appropriately. For the first approach, $\Gamma(3)$ is torsion-free subgroup of index 12 in $\mathrm{PSL}_2(\mathbb{Z})$, so $\mathbb{H}/\Gamma(3)$ is a manifold; a somewhat laborious construction of the fundamental domain and identification of the sides of the fundamental domain gives that the topological type of $\mathbb{H}/\Gamma(3)$ is a sphere with 4 punctures, the Euler characteristic of such a space is -2, so $\chi_{\mathrm{orb}}(\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z}) = -1/6$. For the second approach, if we remove the two bad points from $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$, we obtain a sphere with 3 punctures: this has Euler characteristic -1. Now taking into consideration the two bad points, since they are 0-dimensional simplices we should add on $1/2$ for the bad point corresponding to i and $1/3$ for the bad point corresponding to $\omega = e^{\frac{2\pi i}{3}}$. Hence $\chi_{\mathrm{orb}}(\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})) = -1 + 1/2 + 1/3 = -1/6$.

If one wants $\chi_{\mathrm{orb}}(\mathbb{H}/\mathrm{SL}_2(\mathbb{Z}))$ rather than $\chi_{\mathrm{orb}}(\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z}))$ one wants to take $\chi_{\mathrm{orb}}(\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z}))$ and divide it by 2. This is because the order of the stabilizer of each point in $\mathbb{H}$ under the action of $\mathrm{SL}_2(\mathbb{Z}$ double that order of the stabilizer of each point in $\mathbb{H}$ under the action of $\mathrm{PSL}_2(\mathbb{Z}))$. This leads to the naive calculation $-1+1/2+1/3 = -1/6$ above being replaced by $-1/2+1/4+1/6 = -1/12$. (One can give a precise definition/theorem here too of course). So $\chi_{\mathrm{orb}}(\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})) = -1/12$.

A Quote of Serre

Now we are in a position to start to understand the quote of Serre given immediately below. I believe that is the quote that first introduced me to the subject (when I was

an undergraduate browsing through the library):

> Consider the most obvious discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, namely $G = \mathrm{SL}_2(\mathbb{Z})$. One can compute its "Euler-Poincare characteristic" $\chi(G)$ which turns out to be $-1/12$ (it is not an integer: this is because G has torsion). Now $-1/12$ happens to be the value $\zeta(-1)$ of the Riemann's zeta-function at the point $s = -1$ (a result known already to Euler). And this is not a coincidence! It extends to any totally real number field K, and can be used to study the denominator of $\zeta_K(-1)$. (Better results can be obtained by using modular forms, as was found later.) Such questions are not group theory, nor topology, nor number theory: they are just mathematics. [12]

The first thing to comment on is that Serre refers to the Euler-Poincare characteristic of $\mathrm{SL}_2(\mathbb{Z})$ itself rather than that of $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$. In general, suppose that G that acts properly discontinuously on a contractible manifold X, then *Euler characteristic of G* $\chi(G)$ is defined to be $\chi_{\mathrm{orb}}(X/G)$. This is well defined (independently of X) because if X' is another contractible space then X and X' are homotopy equivalent, so that if $G' < G$ is torsion-free then X/G' and X'/G' are homotopy equivalent and so have the same homology and the same Euler characteristic which implies that $\chi_{\mathrm{orb}}(X/G) = \chi_{\mathrm{orb}}(X'/G)$.

The quantity $\chi(G)$ defined in this way can be shown to be $(1/d)\chi(H)$ where H is a torsion-free finite index subgroup of G and $\chi(H)$ is the alternating sum of the ranks of group cohomology of G. Here group cohomology has a definition in terms of homological algebra (ext and tor) but I don't know homological algebra and so will say no more about this. Regardless, in the papers cited below, $\chi(G)$ is by the definition that I gave above rather than using the language of group cohomology.

The second thing to comment on concerning Serre's quote is the extension of the result that $\zeta(-1) = \chi_{\mathrm{orb}}(\mathrm{SL}_2(\mathbb{Z}))$ to any totally real number field K. This extension is a consequence of three major theorems:

(1) The fact that the Tamagawa number $\tau(\mathrm{SL}_2(O_K)) = 1$.
(2) The functional equation for Dedekind zeta function $\zeta_K(s)$.
(3) A formula of Gauss-Bonnet type due to Harder.

You are familiar with the first two items. The third may be less familiar - let me know if you would like an exposition. Write $n = [K : Q]$, then the diagonal embedding $O_K \hookrightarrow R^n$ induces an embedding $\mathrm{SL}_2(O_K) \hookrightarrow \mathrm{SL}_2(\mathbb{R})^n$ and the Tamagawa number computation (1) gives a formula for the volume[1] of $\mathrm{SL}_2(\mathbb{R})^n/\mathrm{SL}_2(O_K)$ as an infinite product over primes in K which can be recognized to be an elementary factor multiplied by $\zeta_K(2)$. This gives a formula for the volume of the Hilbert modular variety $\mathrm{SO}(2,\mathbb{R})^n\backslash\mathrm{SL}_2(\mathbb{R})^n/\mathrm{SL}_2(O_K)$ in terms of $\zeta_K(2)$. The functional equation (2) allows one to rewrite $\zeta_K(2)$ in the formula in terms of $\zeta_K(-1)$. One can then apply

[1] Here the "volume" is with respect to the Haar measure induced from $\mathbb{R}^{n^2}$ with the Euclidean metric

(3) to replace the volume of the Hilbert modular variety with a suitable multiple of its orbifold Euler characteristic.[2] But $\mathbb{H}^n = \mathrm{SO}(2,\mathbb{R})^n \backslash \mathrm{SL}_2(\mathbb{R})^n$ is a contractible space on which a torsion-free subgroup of finite index subgroup of $\mathrm{PSL}_2(O_K)$ acts freely, one obtains a formula for $\chi(\mathrm{SL}_2(O_K))$ in terms of $\zeta_K(-1)$. The resulting formula is simply

$$\chi(\mathrm{SL}_2(O_K)) = \zeta_K(-1) \tag{0.3}$$

It is remarkable to me that in the course of passing from the simple Tamagawa formula to (0.3) one passes through more complicated formulae (in the course of working through the computations which I have surpressed), but with all of the extra factors introduced eventually cancelling out. I don't have a clear conceptual understanding of why this should be. If the right hand side of (0.3) were more elementary I would say that there should be a more direct proof of (0.3), but it's difficult to imagine what a more direct proof would look like.

A derivation very similar to that of (0.3) shows that for any number field K the following formulae hold:

$$\chi(\mathrm{SL}_n(O_K)) = \zeta_K(-1)\zeta_K(-2)...\zeta_K(1-n) \tag{0.4}$$

$$\chi(\mathrm{Sp}_{2g}(O_K)) = \zeta_K(-1)\zeta_K(-3)\zeta_K(-5)...\zeta_K(1-2g). \tag{0.5}$$

Of course (0.4) simplifies dramatically for $n > 2$ as $\zeta_K(-2) = 0$ and (0.5) simplifies dramatically for K having a nonreal complex embedding since in this case $\zeta_K(-1) = 0$.

As an aside, I believe that the fact that the Tamagawa number attached to a simply connected adelic algebraic group is 1 (and more generally, that Tamagawa numbers in general appear to be uncomplicated rational numbers) is the mathematical phenomenon that has had the greatest psychological impact on me in all of my time thinking about math. There is much I don't know about mathematics and my remark is not one of a well educated and worldly mathematician (not in absolute terms and not relative to a variety of people who you know). However, looking back over the mathematical facts that I was attracted to starting early on in my mathematical development and continuing to this day I see the Tamagawa number as a persistent thread.

[2]The fact χ_{orb} is what appears in the Gauss-Bonnet theorem rather than χ can be taken to be another motivation for introducing the notion χ_{orb}.

Among these facts are:

(1) Euler's formula $\zeta(2) = \pi^2/6$
(2) The Euler product formula
(3) Dirichlet's class number formula
(4) The Birch and Swinnerton-Dyer conjecture
(5) Siegel's product formula in the theory of quadratic forms
(6) Gauss' formula for the number of representations of a natural number as a sum of three squares in terms of the class number of a quadratic field[3]
(7) The hyperbolic volume of the complement of the figure eight knot in the three-sphere can be simply expressed in terms of $\zeta_K(2)$ where $K = \mathbb{Q}(\sqrt{-3})$
(8) Borel's computation of covolumes of the regulators of algebraic K-groups attached to a number ring in terms of special values of the number ring's Dedekind zeta function.

I now see that all of these things hint at or can be proved using Tamagawa numbers. For example, I very recently learned that Bloch [3] gave a formulation of the Birch and Swinnerton-Dyer conjecture in terms of Tamagawa number.

Returning to and concluding my remarks on Serre's comment I will say that a reference for the study of denominators of $\zeta_K(-1)$ where K is totally real using formula (0.3) is [2].

The Harer-Zagier Formula

Now I'm finally able to discuss the Harer-Zagier result. Let $S = S_{g,p}$ be a genus g surface with p marked points. Though many of the results that I state below are independent of p, we will take p to be 1: the marked point will play the role of the basepoint of the fundamental group of the surface. Consider the group G of orientation preserving diffeomorphisms $\phi : S \longrightarrow S$ that fix the marked point. It is natural to consider two such diffeomorphisms to be equivalent if they differ by an isotopy. This amounts to the same as endowing the space of such diffeomorphisms with the compact-open topology and consider two diffeomorphisms to be equivalent if they are in the same connected component. After quotienting G by the aforementioned equivalence relation, one obtains a countable group $\mathrm{Mod}(S)$ called the *mapping class group* of S (or sometimes the *Teichmuller modular group* of S).

Though it may take some time for the uninitiated to see it, $\mathrm{Mod}(S_{g,1})$ is infinite if $g > 0$. If $g = 1$, then $\mathrm{Mod}(S_{g,1})$ is (naturally) isomorphic to $\mathrm{SL}_2(\mathbb{Z})$. For $g > 1$, it is a theorem (first proved by N. Ivanov) that $\mathrm{Mod}(S_{g,1})$ is not isomorphic to an arithmetic lattice in a Lie group. However, for all $g > 0$, $\mathrm{Mod}(S_{g,1})$ does act properly discontinuously on the *Teichmuller space* $T_{g,1}$ attached to S. This is the collection of equivalence classes of complex structures on S under the (unusually strong) notion of equivalence that if M and M' are two endowments of S with complex structure then

[3]this can be thought of as a consequence of the class number formula together with Siegel's product formula

there exists a biholomorphic map from M to M' which identifies the marked points on M and M' respectively and which is *isotopic to the identity map.* Teichmuller space is endowed with a natural topology so that the notion of a properly discontinuous group action makes sense. It is then a theorem that $T_{g,1}$ is homeomorphic to $\mathbb{C}^{3g-2}$ so that we have a group acting properly discontinuously on $\mathbb{R}^n$ for some n, and the quotient meets the above definition of orbifold. The quotient space of $T_{g,1}$ by $\mathrm{Mod}(S_{g,1})$ is $M_{g,1}$, the moduli space of complex structures on S. The space $M_{g,1}$ which has a similar definition as to that of $T_{g,1}$, the only difference being that in the equivalence relation the biholomorphic map from M to M' is not required to be isotopic to the identity, so that one realizes the weaker, more usual notion of equivalence of complex structures on a surface. In the case of $g = 1$, $T_{g,1}$ is the upper half plane and $M_{g,1}$ is the familiar fundamental domain.

In this way we see that $\mathrm{SL}_2(\mathbb{Z})$ has at least two natural generalizations - it can be thought of as being a prototypical arithmetic group, generalizing to $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 2$, to $\mathrm{Sp}_{2n}(\mathbb{Z})$, etc., or it can be thought of as generalizing to $\mathrm{Mod}(S_{g,1})$ for $g \geq 1$. There has been much research in the direction of determining which properties of arithmetic groupshold for $\mathrm{Mod}(S_{g,p})$.

It turns out that $\mathrm{Mod}(S_{g,1})$ has a well defined orbifold Euler characteristic. This is to say that $\mathrm{Mod}(S)$ has a torsion-free finite index subgroup that one can quotient $T_{g,1}$ by, yielding a space which is naturally a manifold. The group $\mathrm{Mod}(S)$ acts on $H_1(S_{p,1}) \cong \mathbb{Z}^{2g}$ so there is a homomorphism $\mathrm{Mod}(S_{g,1}) \longrightarrow \mathrm{GL}_{2g}(\mathbb{Z})$. The image of this homomorphism lies within $\mathrm{SL}_{2g}(\mathbb{Z})$ since each element of $\mathrm{Mod}(S_{g,1})$ is orientation preserving. Any two curves on $S_{g,1}$ can be isotoped so that the number of times that they intersect is minimal - the number of intersections is then called the *intersection number* of the two (isotopy classes) of curves. An element of $\mathrm{Mod}(S_{g,1})$ preserves the intersection number of any two curves, and this leads to the fact that the image of the homomorphism is actually in $\mathrm{Sp}_{2g}(\mathbb{Z})$. One can show directly that the homomorphism surjects onto $\mathrm{Sp}_{2g}(\mathbb{Z})$. Picking a principal congruence subgroup $\Gamma(m)$ of $\mathrm{Sp}_{2g}(\mathbb{Z})$ of level $m \geq 3$ gives a torsion-free subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$. But one can show that nontrivial torsion in $\mathrm{Mod}(S_{g,1})$ gets mapped to nontrivial torsion in $\mathrm{Sp}_{2g}(\mathbb{Z})$, so that the preimage of $\Gamma(m)$ in $\mathrm{Mod}(S_{g,1})$ is a torsion-free finite index subgroup of $\mathrm{Mod}(S_{g,1})$. For the details and other basic facts about the mapping class group the book by Farb and Margalit [5] is very nice.

Knowing that $\chi(\mathrm{Mod}(S_{g,1}))$ is defined, we can now state the theorem of Harer and Zagier [7]:

$$\chi(\mathrm{Mod}(S_{g,1}) = \zeta(1-2g). \tag{0.6}$$

Thus, we see the fact that $\chi(\mathrm{SL}_2(\mathbb{Z})) = -1/12$ does not merely admit generalizations to other arithmetic groups (such as $\mathrm{SL}_2(O_K)$ as Serre's quote alludes to, but it also admits a generalization with $(\mathrm{SL}_2(\mathbb{Z})$ replaced by the mapping class groups of higher genus surfaces. It is possible to use (0.6) to derive formulae for $\chi(S_{g,p})$ for any p, but the formula takes the simplest form when $p = 1$.

The first approach to proving a formula like (0.6) would be by mimicking the proof of (0.3) (adelization of $\mathrm{Mod}(S_{g,1})$, computation of a Tamagawa number, and application of the Gauss-Bonnet theorem, etc.). But because $\mathrm{Mod}(S_{g,1})$ is not an arithmetic lattice in an algebraic group, if $\mathrm{Mod}(S_{g,1})$ possesses an adelization, it doesn't come about in the usual way (by looking at the p-adic points in the equations defining the algebraic group - there are no equations that the elements of the mapping class group are the $\mathbb{Z}$ points of). There is another possible issue in pushing the strategy from the arithmetic case forward in the case of mapping class groups: Harder's Gauss-Bonnet theorem as stated doesn't apply to $M_{g,1}$. Whether or not one easily sees that his proof generalizes is something I have yet to investigate.

I will not say very much about how Harer and Zagier prove the theorem - I don't yet understand well enough to give an exposition that is better than the one in their original paper [7]. In very rough terms, they reduce the problem of computing $\chi(\mathrm{Mod}(S_{g,1})$ to the following combinatorial question: for each $n \geq 2, g > 0$, what is the number of ways $e_g(n)$ to identify pairs of edges of a $2n$-gon to get an orientable genus g surface? In their original paper, Harer and Zagier answer this question by placing the numbers to be determined as coefficients of a generating function and then using novel analytic tools which eventually leads to the evaluation of the coefficients in a form useful for proving (0.6). I think that their proof is deep and opened up new horizons while at the same time falling short of satisfactorily illuminating (0.6). In this respect I would compare it to Euler's evaluation of $\zeta(2n)$ by writing down the Weierstrass product for sine. I know that you have some interest in mathematical physics and so mention in passing that some of the ideas in the original analytic derivation of a formula for $e_g(n)$ were used by Kontsevich in his thesis to prove a conjecture of Witten on the agreement of two models of quantum gravity.

Others later gave shorter derivations of the formula for $e_g(n)$. For example, Zagier gave a short and purely combinatorial derivation of the formula for $e_g(n)$. In the aforementioned paper of Kontsevich, the author gives what seems to me to be a more streamlined analytic derivation of (0.6). My own interest is not so much in a shorter derivation of (0.6) but rather a conceptual derivation. What is short need not be intelligible (c.f. [16]).

The parallel between (0.3) and (0.6) is very strong.

Question 0.1. *Can (0.3) and (0.6) be proved in a uniform fashion?*

Question 0.2. *Can one give a proof of (0.6) that is essentially number theoretic?*

Question 0.3. *What is the connection between (0.6) and the Tamagawa number?*

Since writing my 06/28 message to you, I did have one idea for how one might make progress toward understanding questions 0.2 and 0.3 but it sort of didn't go anywhere. I'm not convinced that it can't be made to work, but it's also too vague for it to be clear that there's anything there. Let X_g be the moduli space of abelian varieties of dimension g. The space X_g is the quotient of the (contractible) Siegel upper half space by $\mathrm{Sp}_{2g}(\mathbb{Z})$, so that one sees that the formula (0.5) is in the case $K = \mathbb{Q}$ is really a reformulation of the formula $\chi_{\mathrm{orb}}(X/G) = \zeta(-1)\zeta(-3)\zeta(-5)...\zeta(1-2g)$. The Torelli

theorem states that the map that takes a Riemann surface with a marked point to its Jacobian variety (taking the marked point to the identity) induces an *injection* from $M_{g,1} \longrightarrow X_g$. This suggests that one may be able to get information about $\chi(\mathrm{Mod}(S_{g,1})) = \chi_{orb}(M_{g,1})$ from that of $\chi(\mathrm{Sp}_{2g}(\mathbb{Z}))$: for example, to rederive (0.6) it suffices to show that $\chi_{\mathrm{orb}}(X_g) = \chi_{\mathrm{orb}}(M_{g,1}) \cdot \chi_{\mathrm{orb}}(X_{g-1})$. But I tried the most obvious things which didn't work and have asked a number of mathematicians what they think and still have no leads.

N. Dunfield suggested to me that $\mathrm{Mod}(S_{g,1})$ may have a notion of "adelization" coming from its profinite completion (as seems to be the case with arithmetic groups) - this is an interesting prospect, although the territory remains murky at the moment.[4]

The question that I raised in my 06/28 message is

Question 0.4. *Do (0.3) and (0.6) have a natural common generalization?*

In order for question 0.4 to have an affirmative answer, it seems that the moduli space $M_{g,1}$ of genus g complex curves with a marked point and the Hilbert modular variety V_K attached to a totally real number field K must have a common generalization. No one who I've talked to seems to be aware of such an object. On a gut level I believe that such a thing ought to exist. If no such object exists then the special value $\zeta_K(-1)$ is "privileged" over $\zeta_K(1-2g)$ for $g > 1$. My understanding is that in the Bloch-Kato conjecture and Beilinson conjectures, the special values of an L-function to the left of the critical strip are qualitatively similar to one another, just as those to the right of the critical strip are qualitatively similar to one another. From this point of view one would not expect $\zeta_K(-1)$ to be privileged. Besides which, (0.6) shows that for $K = \mathbb{Q}$, $\zeta_K(-1)$ is not privileged. On the other hand it appears that it is not so easy to see what the appropriate generalization of $M_{g,1}$ and V_K is if such a generalization does in fact exist.

The most striking image that I take away from the Harer-Zagier formula is that there is some highly nonobvious sense in which $M_{g,1}$ (or $\mathrm{Mod}(S_{g,1})$ if you would like) is attached to $\mathbb{Q}$. This contrasts with the Hilbert modular varieties which are much more obviously attached to K.

I will close my letter with a discussion of two things that I have learned since writing the 06/28 message, which have helped place this material discussed above in context for me.

Euler Characteristics of Groups of Outer Automorphisms of Free Groups

As I discussed above, $\mathrm{SL}_2(\mathbb{Z})$) is the first nontrivial example of two families of groups, namely the arithmetic groups, and the mapping class groups of surfaces. The standard

[4]Added 10/16/09: I have since corresponded with S. Mochizuki who has explained to me that though the profinite completion of an arithmetic group gives the $\mathbb{Z}_p$ points of the group, one really needs the $\mathbb{Q}_p$ points of the group for an adelization and that these come from the Hecke operators which are *not present* in the context of the mapping class group of a surface, as Mochizuki shows in Theorem C of [11]. Mochizuki expressed skepticism as to the possibility of adelizing mapping class groups in light of the absence of Hecke operators. Still, there remains the matter of explaining the Harer-Zagier formula.

perspective in geometric group theory is that $\mathrm{SL}_2(\mathbb{Z})$ is the first of *three* infinite families of groups, the third family being that of outer automorphisms of free groups (to be defined). Since one finds that the Euler characteristics of the mapping class groups have number theoretic significance despite the fact that the mapping class groups of surfaces are almost never arithmetic, it is natural to look for number theoretic significance in the Euler characteristics of the third infinite family of groups.

Given a group G, let $\mathrm{Aut}(G)$ denote the group of all isomorphisms from G to G. Some elements of $\mathrm{Aut}(G)$ arise from conjugation of G by an element of G, these are called inner automorphisms of G and they form a group $\mathrm{Inn}(G)$. The group $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ since if $f : G \longrightarrow G$ is any automorphism and ϕ_a is the inner automorphism given by conjugation by a then the conjugate of ϕ_a by f is $\phi_{f(a)}$. The group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is denoted $\mathrm{Out}(G)$ and is called the outer automorphism group of G.

Groups of outer automorphisms are more familiar than they appear to be to one who has not thought about them before. If $G = \mathbb{Z}^n$ then $\mathrm{Inn}(G)$ is trivial, so $\mathrm{Out(G)} = \mathrm{Aut(G)} = \mathrm{GL}_n(\mathbb{Z})$. I don't know whether other arithmetic groups admit similar interpretation, as a group of outer automorphisms, but it's an interesting thought. For another example, consider the group $\mathrm{Aut}(\pi_1(S_g))$ where S_g is a genus g surface. Then $\mathrm{Mod}(S_{g,0})$ is canonically isomorphic to an index 2 subgroup of $\mathrm{Out}(\pi_1(S_g))$. This is called the theorem of Dehn-Nielsen-Baer. For any p, $\mathrm{Mod}(S_{g,p})$ is canonically isomorphic to a certain well defined subgroup of $\mathrm{Out}(\pi_1(S_{g,p}))$.

Now consider the free group on n generators F_n. The abelianization homomorphism $F_n \longrightarrow \mathbb{Z}^n$ induces a surjective homomorphism $\mathrm{Aut}(F_n) \longrightarrow \mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z}))$ which is trivial on $\mathrm{Inn}(F_n)$ and so gives a surjective homomorphism $\mathrm{Out}(F_n) \longrightarrow \mathrm{GL}_n(\mathbb{Z}))$. For $n = 2$ this homomorphism is an isomorphism. The proof of this last fact is grubby and essentially proceeds by finding a finite presentations of each of the two groups and manipulating the presentations until they are the same. For $n > 2$, the map determined by $x_2 \longrightarrow {x_1}^{-1}x_2x_1, \quad x_3 \longrightarrow {x_2}^{-1}x_3x_2$ is an automorphism, is not an inner automorphism, and becomes trivial in $\mathrm{GL}_n(\mathbb{Z})$, so the homomorphism is not injective for $n > 2$.

In [4], Vogtmann and Culler constructed a space X_n on which $\mathrm{Out}(F_n)$ naturally acts and proved that X_n is contractible. I don't know whether this space is homeomorphic to some Euclidean space, regardless one can still define orbifold Euler characteristic of the quotient by appropriately modifying the ad hoc definition that I gave: the key thing is really that the space is contractible and that the group $\mathrm{Out}(F_n)$ has a torsion-free finite index subgroup. The kernel of the homomorphism $\mathrm{Out}(F_n) \longrightarrow \mathrm{GL}_n(\mathbb{Z})$ is torsion-free [1], so that the inverse image of a finite index torsion-free subgroup of $\mathrm{GL}_n(\mathbb{Z})$ is torsion-free and finite index in $\mathrm{Out}(F_n)$, and the Euler characteristic of $\mathrm{Out}(F_n)$ is defined. The parallel between $\mathrm{Mod}(S_{g,s})$ and $\mathrm{Out}(F_n)$ is clear.

In two subsequent papers [13, 14], Smillie and Vogtmann gave a generating function for the Euler characteristics of $\mathrm{Out}(F_n)$ as n varies, and found that the denominators of these numbers obey analogs of some of the divisibility properties of the denominators of the values $\zeta(1-2g)$, g a positive integer. Specifically, Theorem 5.2 of [4] reads: "p is an odd prime and p divides the denominator of $\chi(\mathrm{Out}(F_n))$ then $n \geq p-1$. If

$n = p - 1$, then p divides the denominator of $\chi(\mathrm{Out}(F_n))$ exactly once." Theorem 4.3 of [14] gives a more precise result.

In the second [13] of the two papers the authors say, "The divisibility properties of numerators of Bernoulli numbers have a number theoretic interpretation relating to regular primes.... It would be interesting to know whether the divisibility properties of numerators of $\chi(\mathrm{Out}(F_n))$ have a similar interpretation."

Here I am rather less confident than there is something of number theoretic significance lurking beneath the surface than in the case of (0.6), but if there were, it could be quite interesting...

Differentiable Structures on Spheres

Two months ago I happened across an inspirational video of a lecture by Barry Mazur [10]. The subject of the lecture is the Kervaire-Milnor formula for the number $|\theta_{4n-1}|$ of equivalence classes of differentiable manifolds which are homeomorphic to the $4n - 1$ sphere, where the equivalence relation is diffeomorphism. That $|\theta_{4n-1}| \neq 1$ for some n (specifically for $n = 2$) is John Milnor's famous discovery from the 1950's: his discovery of so called "exotic spheres." The formula reads

$$|\theta_{4n-1}| = E_n |\Pi_{4n-1}| |\zeta(1 - 2n)| \quad n \geq 1. \tag{0.7}$$

Here $E_n = (2^{2n-3})(2^{2n-1} - 1)$ for n even and twice this for n odd, and $|\Pi_{4n-1}|$ is a certain integer appearing in homotopy theory which is sufficiently robust in prime factors so that the right hand side is a natural number (as it must be for the formula to hold). The formula (0.7) has no logical relation to the rest of this letter. However, there is a *thematic* relation to the rest of this letter - the appearance of $\zeta(1 - 2n)$ can be interpreted as signifying that the collection of $4n - 1$ spheres is "attached to $\mathbb{Q}$" in a highly nonobvious way. This raises

Question 0.5. *Let $n \geq 1$ and K be a number field. Does the Kervaire-Milnor formula admit a natural generalization with the $4n - 1$ sphere replaced by a suitable space and $\zeta(1 - 2n)$ replaced by $\zeta_K(1 - 2n)$?*

Conclusion

Like Y. Manin [9], I find the prospect of number theory connecting up with physics in a deep way to be strongly appealing. More generally, I dream of visceral manifestations of number theoretic phenomena in naturally occuring spaces. The formulae (0.6) and (0.7) hint at the possibility that my dreams may eventually be realized and that (for example) your work on functoriality may ultimately have ramifications in more varied contexts than would meet the eye of a casual observer.

Warmly and with admiration,

Jonah S.

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